

Coherent state quantization of paragrassmann algebras

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Abstract

By using a coherent state quantization of paragrassmann variables, operators are constructed in finite Hilbert spaces. We thus obtain in a straightforward way a matrix representation of the paragrassmann algebra. This algebra of finite matrices realizes a deformed Weyl-Heisenberg algebra. The study of mean values in coherent states of some of these operators leads to interesting conclusions.

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1 Introduction

In this paper, we present the quantization of a paragrassmann algebra through appropriate coherent states [1]. paragrassmann algebras were introduced mainly within the context of parastatistics, fractional statistics and more. For a quite comprehensive review of the mathematics involved in such a structure, we refer for instance to [2, 3]. Generalizations of Grassmann algebras have been very popular during the nineties (see e.g. [4]-[14] and references therein) from different points of view. Those generalizations were mainly motivated by searches of exact solutions or at least integrability properties found in $2d$ conformal field theories, anyonic models and topological field theories which led to unusual statistics (parastatistics, fractional, braid statistics, ...). There were also attempts to generalize supersymmetry to parasupersymmetry.

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In Section 2, we explain the principle and the basics of coherent state construction on an arbitrary measure space and the issued quantization, which we call coherent state (CS) quantization throughout this paper. CS quantization extends to a wider set of situations the Klauder-Berezin-Toeplitz (called also anti-Wick) quantization [15, 16, 17]. In Section 3 we recall the main features of the one-variable paragrassmann algebra equipped with an integral calculus in the sense of Berezin-Majid [18, 14]. In Section 4, we build the corresponding coherent states as they have been introduced in [20, 22, 21, 19]. In Section 5, we proceed with the quantization of the one-variable paragrassmann algebra through these coherent states and describe in detail the matrix algebra obtained from this procedure. In Section 6, we examine the *dequantization* that is implemented via the mean values of matrices in coherent states (“lower symbols”) and which leads to the original paragrassmann algebra. In Section 9, we generalize the procedure to d -variable paragrassmann algebra. Eventually, we give in the conclusion some insights on the use of such a formalism in finite quantum mechanics and possibly in quantum information.

2 Coherent state quantization

Let $X = \{x \mid x \in X\}$ be a set equipped with a measure $\mu(dx)$ and $L^2(X, \mu)$ the Hilbert space of square integrable functions $f(x)$ on X :

$$\|f\|^2 = \int_X |f(x)|^2 \mu(dx) < \infty, \quad \langle f_1 | f_2 \rangle = \int_X \overline{f_1(x)} f_2(x) \mu(dx). \quad (1)$$

Let us select, among elements of $L^2(X, \mu)$, an orthonormal set $\mathcal{S}_N = \{\phi_n(x)\}_{n=0}^{N-1}$, N being finite or infinite, which spans, by definition, a separable Hilbert subspace \mathcal{K} in $L^2(X, \mu)$. We demand this set to obey the following crucial condition

$$0 < \mathcal{N}(x) \equiv \sum_n |\phi_n(x)|^2 < \infty \text{ almost everywhere.} \quad (2)$$

This condition is obviously trivial for finite N . Let \mathcal{H} be a separable Hilbert space with orthonormal basis $\{|e_n\rangle, n = 0, 1, \dots, N-1\}$ in one-to-one correspondence with the elements of the set \mathcal{S}_N . Then consider the family of states $\{|x\rangle, x \in X\}$ in \mathcal{H} obtained through the following linear superpositions:

$$|x\rangle \equiv \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_n \overline{\phi_n(x)} |e_n\rangle. \quad (3)$$

This defines an injective map (which should be continuous w.r.t some minimal topology affected to X for which the latter is locally compact):

$$X \ni x \mapsto |x\rangle \in \mathcal{H},$$

By construction, these *coherent* states obey

- **Normalization**

$$\langle x | x \rangle = 1, \quad (4)$$

- **Resolution of the unity in \mathcal{H}**

$$\int_X |x\rangle\langle x| \mathcal{N}(x) \mu(dx) = \mathbb{I}_{\mathcal{H}}, \quad (5)$$

A *classical* observable is a function $f(x)$ on X having specific properties, depending on supplementary structure allocated to set X . Its coherent state quantization consists in associating to $f(x)$ the operator

$$A_f := \int_X f(x) |x\rangle\langle x| \mathcal{N}(x) \mu(dx). \quad (6)$$

Operator A_f is symmetric if $f(x)$ is real-valued, and is bounded if $f(x)$ is bounded. The original $f(x)$ is an “upper symbol” in the sense of Lieb [17] or a contravariant symbol in the sense of Berezin [15] for the usually non-unique operator A_f . It will be called a *classical* observable if its “lower symbol” in the sense of Lieb [17] or its covariant symbol in the sense of Berezin [15], $\check{A}_f(\alpha) \stackrel{\text{def}}{=} \langle x | A_f | x \rangle$, has mild functional properties to be made precise (e.g. is a smooth function) according to further topological properties given to the original set X (e.g. is a symplectic manifold).

Such a quantization of the set X is in one-to-one correspondence with the choice of the frame of coherent states encoded by the resolution of the unity (5). To a certain extent, a quantization scheme consists in adopting a certain point of view in dealing with X (compare with Fourier or wavelet analysis in signal processing). Here, the validity of a precise frame choice is asserted by comparing spectral characteristics of quantum observables A_f with data provided by some specific protocol in the observation of X .

3 The essential of paragrassmann algebra

In this paper, we consider as an observation set X the paragrassmann algebra Σ_k [2, 4, 14, 20]. We recall that a Grassmann (or exterior) algebra of a given vector space V over a field is the algebra generated by the exterior (or wedge) product for which all elements are nilpotent, $\theta^2 = 0$. paragrassmann algebras are generalizations for which, given an integer $k > 2$, all elements obey $\theta^{k'} = 0$, where $k' = k$ for odd k and $k' = k/2$ for even k . For a given k , we define Σ_k as the linear span of $\{1, \dots, \theta^n, \dots, \theta^{k'-1}\}$ and of their respective *conjugates* $\bar{\theta}^n$: here θ is a paragrassmann variable satisfying $\theta^{k'} = 0$.

Variables θ and $\bar{\theta}$ do not commute:

$$\theta \bar{\theta} = q_k \bar{\theta} \theta, \quad (7)$$

where $q_k = q = e^{\frac{2\pi i}{k}}$ for odd k , and $q_k = q^2$ for even k . The motivation backing the above definition is to choose the deformation parameter in the paragrassman algebra as a root of the unity of the same order as the nilpotency order: $q^k = 1$ for odd k , and $(q^2)^{k/2} = 1$ for even k .

Thus, the number k fixes the order of the root q and the order of nilpotency of the paragrassmann algebra. The distinction between even and odd values of k regarding the nilpotency order is necessary to enforce that the Fock representations of the quantized paragrassmann algebras have dimension k' , and therefore match the representation theory of the q -oscillator algebra when q is a k th root of unity (see, e.g., [23]).

A measure on X is defined as

$$\mu(d\theta d\bar{\theta}) = d\theta w(\theta, \bar{\theta}) d\bar{\theta}. \quad (8)$$

The integral over $d\theta$ and $d\bar{\theta}$ should be understood in the sense of a Berezin-Majid-Rodríguez-Plaza integral [14]:

$$\int d\theta \theta^n = 0 = \int d\bar{\theta} \bar{\theta}^n, \quad \text{for } n = 0, 1, \dots, k' - 2, \quad (9)$$

with

$$\int d\theta \theta^{k'-1} = 1 = \int d\bar{\theta} \bar{\theta}^{k'-1}. \quad (10)$$

The “weight” $w(\theta, \bar{\theta})$ is given by the q -deformed polynomial

$$w(\theta, \bar{\theta}) = \sum_{n=0}^{k'-1} [n]_q! \theta^{k'-1-n} \bar{\theta}^{k'-1-n}. \quad (11)$$

We adopt in this paper the “symmetric” definition¹ for q -deformed numbers which makes them real:

$$[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sin \frac{2\pi x}{k}}{\sin \frac{2\pi}{k}}. \quad (13)$$

The q -factorial function is defined by

$$[n]_q! = [1]_q [2]_q \cdots [n]_q, \quad \text{with} \quad [0]_q! = 1. \quad (14)$$

The measure (8) allows one to define a complex pseudo-Hilbertian structure on the vector space that is the k^2 -dimensional linear span on \mathbb{C} of all monomials of the type $\theta^m \bar{\theta}^{\bar{m}}$ for

¹In many places, the q -deformed numbers are defined in asymmetric way as :

$$[x]_q := \frac{1 - q^x}{1 - q}, \quad (12)$$

and so are complex.

$m, \bar{m} = 0, 1, \dots, k' - 1$. A generic element of this algebra, element of the pseudo-Hilbert space $L_w^2 \equiv L^2(X, \mu(d\theta d\bar{\theta}))$ will be denoted:

$$v(\theta, \bar{\theta}) = \sum_{m, \bar{m}} v_{m\bar{m}} \theta^m \bar{\theta}^{\bar{m}}, \quad v_{m\bar{m}} \in \mathbb{C}. \quad (15)$$

Consistently, the inner product in L_w^2 is defined by

$$(v, v') = \int d\theta : \overline{v(\theta, \bar{\theta})} v'(\theta, \bar{\theta}) w(\theta, \bar{\theta}) : d\bar{\theta}, \quad (16)$$

where the symbol “ $: \cdot :$ ” means an antinormal ordering, i.e., all nonbarred θ stand on the left. In general, the monomials $\theta^m \bar{\theta}^{\bar{m}}$ are not orthogonal. Only pairs $(\theta^m \bar{\theta}^{\bar{m}}, \theta^{m'} \bar{\theta}^{\bar{m}'})$ for which $m - \bar{m} \neq m' - \bar{m}'$, are orthogonal. Note that we have the same occurrence in the Hilbert space $L^2(\mathbb{C}, e^{-|z|^2} d^2z/\pi)$ with monomials $z^m \bar{z}^{\bar{m}}$. In the latter case, an orthonormal basis is provided by complex Hermite polynomials [24, 25]. An adequate q -deformation of these complex Hermite polynomials could provide an orthonormal basis for $L_w^2 \equiv L^2(X, \mu(d\theta d\bar{\theta}))$.

Thanks to the inner product in (16), we can define a pseudo-norm:

$$\|v(\theta, \bar{\theta})\|^2 = \iint d\theta : \overline{v(\theta, \bar{\theta})} v(\theta, \bar{\theta}) w(\theta, \bar{\theta}) : d\bar{\theta}. \quad (17)$$

As expected, this quantity is not positively defined on the entire pseudo-Hilbert space L_w^2 ; on the other hand, it is positively defined on the linear span of powers of θ and $\bar{\theta}$ separately; i.e. on functions of the form

$$f(\theta, \bar{\theta}) = a_0 + \sum_{n=1}^{k'-1} a_n \theta^n + \sum_{n=1}^{k'-1} b_n \bar{\theta}^n. \quad (18)$$

Consequently a distance and a metric topology can also be defined on such a subspace.

4 k -fermionic coherent states

In order to implement the CS quantization for the paragrassmann algebra Σ_k we need first to build a family of coherent states. For that purpose, we follow the scheme described in Section 2. Inspired by [19, 20] we choose as an orthonormal set in L_w^2 the following monomial functions:

$$\phi_n(\theta, \bar{\theta}) = \frac{\bar{\theta}^n}{([n]_q!)^{\frac{1}{2}}}, \quad n = 0, 1, \dots, k' - 1. \quad (19)$$

The (nonnormalized) *paragrassmann* or *k-fermionic* coherent states [2, 3, 20, 22, 21] should be understood as elements of $\Sigma_k \otimes \mathbb{C}^{k'}$. They read as (we omit the tensor product symbol):

$$|\theta\rangle = \sum_{n=0}^{k'-1} \overline{\phi_n(\theta, \bar{\theta})} |n\rangle = \sum_{n=0}^{k'-1} \frac{\theta^n}{([n]_q!)^{\frac{1}{2}}} |n\rangle, \quad (20)$$

where $\{|n\rangle, n = 0, 1, \dots, k' - 1\}$ is an orthonormal basis of the Hermitian space $\mathbb{C}^{k'}$, e.g. the canonical basis.

The resolution of the unity I_k in $\mathbb{C}^{k'}$ follows automatically from the orthonormality of the set (19):

$$\int \int d\theta |\theta\rangle w(\theta, \bar{\theta}) (\theta| d\bar{\theta} = I_k. \quad (21)$$

The weight function is given in (11).

5 Quantization with *k*-fermionic coherent states

We now have all the ingredients needed to quantize the set $X = \Sigma_k$ of paragrassmann variables along the scheme described in section 2. The quantization of a paragrassmann-valued function $f(\theta, \bar{\theta})$ maps f to the linear operator A_f on $\mathbb{C}^{k'}$:

$$A_f = \int \int d\theta |\theta\rangle f(\theta, \bar{\theta}) w(\theta, \bar{\theta}) (\theta| d\bar{\theta}, \quad (22)$$

with no consideration of ordering, at the moment.

5.1 Some relations for coherent state quantization of paragrassmann algebras

Because of the noncommutativity (7) between paragrassmann variables, the quantization proposed in (22) is not straightforward and one should define an order in which the elements in the integral should be written. In the following are presented some possibilities of how to achieve this goal while quantizing a general function of θ and $\bar{\theta}$.

$$f(\theta, \bar{\theta}) \mapsto A_f = \int d\theta |\theta\rangle : f(\theta, \bar{\theta}) w(\theta, \bar{\theta}) : (\theta| d\bar{\theta}, \quad (23)$$

where the symbol “ $: \dots :$ ” stands for antinormal ordering already adopted in (16).

$$\begin{aligned} f(\theta, \bar{\theta}) \stackrel{L}{\mapsto} A_f^L &= \int d\theta |\theta\rangle f(\theta, \bar{\theta}) w(\theta, \bar{\theta}) (\theta| d\bar{\theta}, \\ f(\theta, \bar{\theta}) \stackrel{R}{\mapsto} A_f^R &= \int d\theta |\theta\rangle w(\theta, \bar{\theta}) f(\theta, \bar{\theta}) (\theta| d\bar{\theta}. \end{aligned} \quad (24)$$

Adopting a certain ordering may lead to different quantizations, for instance to the appearance of an extra q -dependent coefficient inside the summations like we have below for the quantized versions of θ and $\bar{\theta}$:

$$\begin{aligned} A_\theta &= A_\theta^L = \sum_{n=0}^{k'-1} \left([n+1]_q \right)^{1/2} |n\rangle \langle n+1|, \\ A_\theta^R &= \sum_{n=0}^{k'-1} \left([n+1]_q \right)^{1/2} q_k^{n+2} |n\rangle \langle n+1|, \\ A_{\bar{\theta}} &= A_{\bar{\theta}}^R = \sum_{n=0}^{k'-1} \left([n+1]_q \right)^{1/2} |n+1\rangle \langle n|, \\ A_{\bar{\theta}}^L &= \sum_{n=0}^{k'-1} \left([n+1]_q \right)^{1/2} q_k^{n+2} |n+1\rangle \langle n|. \end{aligned}$$

Note that, for $f(\theta)$, $g(\theta)$, polynomials in θ , the following algebra homomorphism holds,

$$A_{fg}^L = A_f^L A_g^L = A_{fg} = A_f A_g. \quad (25)$$

Likewise, for $f(\bar{\theta})$ and $g(\bar{\theta})$ polynomials in $\bar{\theta}$, one has

$$A_{fg}^R = A_f^R A_g^R = A_{fg} = A_f A_g.$$

In the following we mainly use the antinormal ordering, already adopted in Equation (16). Consequently, we adopt the rule that θ is quantized as $A_\theta = A_\theta^L$ and $\bar{\theta}$ as $A_{\bar{\theta}} = A_{\bar{\theta}}^R$.

Now we consider higher order polynomials in θ and $\bar{\theta}$. One can easily check that

$$A_\theta^2 = A_{\theta^2}, \quad A_\theta^3 = A_{\theta^3},$$

and so on up to order $n = k' - 1$. In order to prove this, let us show that $A_{\theta^n} A_\theta = A_{\theta^{n+1}}$:

$$\begin{aligned} A_{\theta^n} A_\theta &= \sum_{m,m'=0}^{k'-1} \left\{ \frac{[m+n]_q!}{[m]_q!} [m'+1]_q \right\}^{1/2} |m\rangle \langle m+n |m'\rangle \langle m'+1| \\ &= \sum_{m=0}^{k'-1} \left\{ \frac{[m+n+1]_q!}{[m]_q!} \right\}^{1/2} |m\rangle \langle m+n+1| = A_{\theta^{n+1}}. \end{aligned}$$

This can also be viewed as a direct consequence of the algebra homomorphism defined in (25); one then ends with the interesting result $A_\theta^n = A_{\theta^n}$. Similarly, one has $A_{\bar{\theta}}^n = A_{\bar{\theta}^n}$. These properties ensure that the nilpotency of the paragrassmann variables is preserved after quantization.

Calculations regarding the quantization of higher order mixed terms of θ and $\bar{\theta}$ are also given in the appendix.

Hence, we recover the $k' \times k'$ -matrix realization of the so-called k' -fermionic algebra $F_{k'}$ [20, 22] (see appendix).

In the simplest case where one quantizes the product $\theta\bar{\theta}$ one gets the interesting formulas:

$$\begin{aligned} A_{\theta\bar{\theta}} &= \sum_{n=0}^{k'-1} \left| [n+1]_q \right| |n\rangle \langle n| \\ &= A_\theta^L A_{\bar{\theta}}^R = A_\theta A_{\bar{\theta}}. \end{aligned} \quad (26)$$

However

$$A_{\bar{\theta}\theta} \neq A_{\bar{\theta}} A_\theta; \quad (27)$$

this is a central point in the discussion that follows.

5.2 Discussion of the quantized algebra

The quantization of the q -commutation relation (7) then is given by:

$$A_\theta A_{\bar{\theta}} - q A_{\bar{\theta}} A_\theta = \overline{Q} \quad (28)$$

where the matrix \overline{Q} is defined by

$$\overline{Q} = \bar{q}^N = q^{-N} = \sum_{n=0}^{k'-1} \bar{q}^n |n\rangle \langle n|$$

Equation (28) is nothing but the defining relation for the Biedenharn-Macfarlane oscillator [26, 27]. Another popular form for equation (28) can be obtained by defining $B_\theta = q^{N/2} A_\theta$ and $B_{\bar{\theta}} = B_{\bar{\theta}} q^{N/2}$:

$$B_\theta B_{\bar{\theta}} - q^2 B_{\bar{\theta}} B_\theta = 1.$$

In addition, the following rule holds

$$A_\theta A_{\bar{\theta}} - \bar{q} A_{\bar{\theta}} A_\theta = Q, \quad (29)$$

with

$$Q = q^N = \sum_{n=0}^{k'-1} q^n |n\rangle \langle n|.$$

The fact that equations (28) and (29) hold simultaneously is necessary to ensure that the operators A_θ and $A_{\bar{\theta}}$ are hermitian conjugate to each other. This actually proved useful while constructing paragrassman coherent states in [22]. And even though it was an *ad hoc* supposition then, the present result justifies that choice.

In addition, the fact that $A_\theta^n = A_{\theta^n}$ and $A_{\bar{\theta}}^n = A_{\bar{\theta}^n}$ ensures the nilpotency of the corresponding quantized operators. This fact was also used in [21, 22] where the constructed paragrassmann coherent states were associated to nilpotent representations of Biedenharn-Macfarlane deformed oscillator.

The q -commutation relation of the “classical” variables θ and $\bar{\theta}$:

$$\theta\bar{\theta} - q_k\bar{\theta}\theta = 0$$

is replaced by a *non-q-commutativity* relation (28) (or (29)) between the corresponding quantum operators A_θ and $A_{\bar{\theta}}$. This is in complete analogy with the (canonical or coherent states or something else) quantization of the usual harmonic oscillator where the commutativity of the variables, *position* and *momentum* is replaced by a noncommutativity of the corresponding operators in quantum physics.

It is important to note at this stage that these results justify the choices in works such as [22] where the structure of paragrassmann algebras was associated with that of the Macfarlane-Biedenharn oscillator.

However, if a different definition of the q -numbers (13) were adopted one would get the same results, except for those presented in section 5.2, where in fact different deformations of the usual harmonic oscillator are found depending on the definition adopted for the q -numbers (13). The common point between the results is that paragrassmann coherent states are associated to nilpotent representations of deformations of the harmonic oscillator.

6 Lower symbols and “classical limit”

The lower symbols for A_θ and $A_{\bar{\theta}}$ are given by the expressions

$$\begin{aligned} (\theta| A_\theta | \theta) &= (\theta | \theta) \theta, \\ (\theta| A_{\bar{\theta}} | \theta) &= \bar{\theta} (\theta | \theta). \end{aligned}$$

So, if one can formally define

$$\langle A_\theta \rangle \equiv \frac{(\theta| A_\theta | \theta)}{(\theta | \theta)} = \theta, \quad \langle A_{\bar{\theta}} \rangle = \frac{(\theta| A_{\bar{\theta}} | \theta)}{(\theta | \theta)} = \bar{\theta}, \quad (30)$$

then the normalized lower symbols satisfy the classical relation

$$\langle A_\theta \rangle \langle A_{\bar{\theta}} \rangle = q_k \langle A_{\bar{\theta}} \rangle \langle A_\theta \rangle.$$

This secures some sort of a returning path, allowing one to recover the classical algebra from the quantized one using the lower symbols (30).

Some useful relations are given in what follows

$$\begin{aligned}\theta(\theta|\theta) &= (\theta|q_k\theta)\theta, \quad (\theta|\theta)\theta = \theta(q_k\theta|\theta), \\ \bar{\theta}(\theta|\theta) &= (q_k\theta|\theta)\bar{\theta}, \quad (\theta|\theta)\bar{\theta} = \bar{\theta}(\theta|q_k\theta), \\ (\theta|\theta) &= (q_k\theta|q_k\theta) = \sum_{n=0}^{k'-1} \frac{(\bar{\theta})^n (\theta)^n}{[n]_q!} = \sum_{n=0}^{k'-1} \bar{q}_k^{n^2} \frac{(\theta)^n (\bar{\theta})^n}{[n]_q!}.\end{aligned}$$

The lower symbol of an arbitrary linear operator A in $\mathbb{C}^{k'}$, defined by the matrix $(a_{mm'})$ as $A = \sum_{mm'} a_{mm'} |m\rangle \langle m'|$ is given by the general element of the algebra \sum_k :

$$(\theta|A|\theta) = \sum_{n\bar{n}} \frac{a_{\bar{n}n}}{\left([n]_q! [n]_q!\right)^{1/2}} (\bar{\theta})^{\bar{n}} (\theta)^n = \sum_{n\bar{n}} \frac{\bar{q}_k^{n\bar{n}} a_{\bar{n}n}}{\left([n]_q! [n]_q!\right)^{1/2}} (\theta)^n (\bar{\theta})^{\bar{n}}. \quad (31)$$

7 Upper symbols and Moyal product

The question now is to determine, for an arbitrary linear operator A in $\mathbb{C}^{k'}$, if there exists $f \in L_w^2$ such that $A = A_f$, and whether that f is unique or not. It suffices to show that the coefficients a_{mn} of A_f , $A_f = \sum_{mn} a_{mn} |m\rangle \langle n|$, are uniquely determined by those of $f(\theta, \bar{\theta}) = \sum_{st} f_{st} \theta^s \bar{\theta}^t$ and *vice versa*. After performing the Grassmann integrations in the general expression of the correspondence relation (22), one is left with the following relations between the coefficients of f and A_f :

$$a_{nn'} = \sum_s M_{nn',s} f_{s,n-n'+s}, \quad (32)$$

where

$$M_{nn',s} = \left(\frac{[n+s]_q! [n+s]_q!}{[n]_q! [n']_q!} \right)^{1/2}. \quad (33)$$

In order to show that M is invertible, we distinguish two cases, namely, $n' + p$ with non-negative $p \geq 0$ and non-positive $0 \geq p$. Since the latter is not independent from the former, we concentrate on the $p \geq 0$ case. For $p \geq 0$, we write (32) as

$$a_n^{(p)} = \sum_{s=0}^{k'-1-n} M_{n,s}^{(p)} f_s^{(p)},$$

with the obvious identifications $a_n^{(p)} \equiv a_{n,n-p}$, $M_{n,s}^{(p)} = M_{n,n-p,s}$ and $f_s^{(p)} = f_{s,p+s}$. The upper summation limit arises from the restriction in $M_{n,s}^{(p)}$ that $n + s < k'$ and $n \geq p$. The

linear system thus obtained can be better understood in terms of the matrices

$$\begin{pmatrix} a_{k'-1}^{(p)} \\ a_{k'-2}^{(p)} \\ \vdots \\ a_p^{(p)} \end{pmatrix} = \begin{pmatrix} M_{k'-1,0}^{(p)} & 0 & \cdots & 0 \\ M_{k'-2,0}^{(p)} & M_{k'-2,1}^{(p)} & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ M_{p,0}^{(p)} & M_{p,1}^{(p)} & \cdots & M_{p,k'-1-p}^{(p)} \end{pmatrix} \begin{pmatrix} f_0^{(p)} \\ f_1^{(p)} \\ \vdots \\ f_{k'-1-p}^{(p)} \end{pmatrix}.$$

Note that for each integer s in $k' - 1 - p \geq s \geq 0$, the coefficient $M_{k'-1-s,s}^{(p)}$ of $f_s^{(p)}$ in $a_{k'-1-s}^{(p)}$ is nonzero. This means that the above triangular matrix can be inverted, since its determinant is $\prod_{s=0}^{k'-1-p} M_{k'-1-s,s}^{(p)} \neq 0$. The case with $p < 0$ leads to an analogous system with identical non-singular triangular matrix consisting of the coefficients (33).

Therefore, we have established that the correspondence $f \mapsto A_f$ is invertible. For instance, for $k = 4$, the inverse image of any order 2 matrix a_{mn} is

$$f(\theta, \bar{\theta}) = a_{11} + a_{01}\theta + a_{10}\bar{\theta} + (a_{00} - a_{11})\theta\bar{\theta}. \quad (34)$$

As a result of the unique correspondence $f \mapsto A_f$, the algebra of matrices on $\mathbb{C}^{k'}$ is isomorphic to the algebra of functions $f(\theta, \bar{\theta})$ with Moyal multiplication:

$$A_f A_g = A_{f \star g}.$$

As an example, let us consider the $k = 4$ case and an application to the complex quaternion algebra. Given $f \mapsto A_f$ and $g \mapsto A_g$, the Moyal product of the symbols f and g of the operators A_f and A_g is

$$\begin{aligned} f \star g(\theta, \bar{\theta}) &= f_{01}g_{10} + f_{00}g_{00} + ((f_{00} + f_{11})g_{10} + f_{10}g_{00})\theta + (f_{01}(g_{00} + g_{11}) + f_{00}g_{01})\bar{\theta} \\ &\quad + (f_{00}g_{11} + f_{11}(g_{00} + g_{11}) + f_{10}g_{01} - f_{01}g_{10})\theta\bar{\theta}. \end{aligned} \quad (35)$$

One can easily see that $\theta \star \theta = \bar{\theta} \star \bar{\theta} = 0$. Now consider the (Pauli) matrix representation for the complex quaternions,

$$Z = z_0 + z_1(i\sigma^1) + z_2(-i\sigma^2) + z_3(i\sigma^3), \quad z_i \in \mathbb{C}.$$

One can use formulae (34,35) to compute the lower symbol of the product ZW , where Z and W are as above. As expected, the Moyal product reproduces the quaternion multiplication law in symbol space, for the complex coefficients x_i defining $z \star w$ are

$$x_0 = z_0w_0 - \mathbf{z} \cdot \mathbf{w}, \quad \mathbf{x} = z_0\mathbf{w} + w_0\mathbf{z} + \mathbf{z} \times \mathbf{w}.$$

Equivalently, one can compute the symbols $i\sigma^1 \mapsto I$, $-i\sigma^2 \mapsto J$ and $i\sigma^3 \mapsto K$ and easily see that

$$\begin{aligned} I \star I &= J \star J = K \star K = -1, \\ I \star J &= -J \star I = K \end{aligned}$$

. Thus, the general symbol according to (34) is

$$\begin{aligned} z(\theta, \bar{\theta}) &= z_0 - iz_3 + (iz_1 - z_2)\theta + (iz_1 + z_2)\bar{\theta} + 2iz_3\theta\bar{\theta} \\ &= z_0 + z_1I + z_2J + z_3K. \end{aligned}$$

from which is evident that $z \star w$ is quaternion multiplication.

8 Paragrassmannian Fock-Bargmann representation of operators

To any vector $|\psi\rangle$ in $\mathbb{C}^{k'}$ let us associate the function $\psi(\theta)$ in L_w^2 defined by

$$\mathcal{W} : |\psi\rangle \mapsto \psi(\theta) \stackrel{\text{def}}{=} (\theta|\psi\rangle = \sum_{n=0}^{k'-1} \frac{\theta^n}{([n]_q!)^{\frac{1}{2}}} \psi_n, \quad \psi_n \equiv \langle n|\psi\rangle, \quad (36)$$

This yields an isometry between $\mathbb{C}^{k'}$ and the Hilbert subspace of L_w^2 spanned by the first k powers of θ . Under such an isometry, the operators A_θ and $A_{\bar{\theta}}$ are denoted respectively by:

$$\partial_\theta \stackrel{\text{def}}{=} \mathcal{W}A_\theta\mathcal{W}^\dagger \quad \mathfrak{m}_\theta \stackrel{\text{def}}{=} \mathcal{W}A_{\bar{\theta}}\mathcal{W}^\dagger. \quad (37)$$

They realize themselves, respectively, as paragrassmannian derivative and multiplication as:

$$\partial_\theta \theta^n = [n]_q \theta^{n-1}, \quad \mathfrak{m}_\theta f(\theta) = \theta f(\theta). \quad (38)$$

Hence, this “Fock-Bargmann” representation of operators allows one to recover in a quite natural way the algebra of type Π_k with two nilpotent generators θ and ∂ as they are described in [3].

9 Generalization to d -dimensional paragrassmann variable algebra

All the results obtained so far can be generalized to the d -dimensional case for which the observation set, X , is given by a d -dimensional paragrassmann algebra.

A d -dimensional paragrassmann algebra, [14], is generated by d paragrassmann variables θ_i and their respective conjugates $\bar{\theta}_i$ where $i = 1, 2, \dots, d$. The variables do not commute with each other. One has instead the following q -commutation relations:

$$\begin{aligned} \theta_i \theta_j &= q_k \theta_j \theta_i, \\ \bar{\theta}_i \bar{\theta}_j, &= q_k \bar{\theta}_j \bar{\theta}_i, \quad i, j = 1, 2, \dots, d, \quad i < j. \\ \theta_i \bar{\theta}_j &= \bar{q}_k \bar{\theta}_j \theta_i. \end{aligned} \quad (39)$$

Here q_k is a k th root of unity for odd k and a $\frac{k}{2}$ th root of unity for even k , as before. All these variables are nilpotent: $\theta_i^{k'} = \bar{\theta}_i^{k'} = 0$. Relations (39) can be written in a compact form as follows

$$\alpha_i \beta_j = q_k^{ab} \beta_j \alpha_i \quad i < j, \quad a, b \in \{-1, 1\}. \quad (40)$$

A measure on the observation set, analogous to the one in (8), is given by:

$$\mu(d\theta d\bar{\theta}) = d\theta_d \dots d\theta_1 w(\theta, \bar{\theta}) d\bar{\theta}_d \dots d\bar{\theta}_1. \quad (41)$$

We use here the short-handed notation $w(\theta, \bar{\theta})$ for the weight, which is in fact a polynomial function of all the variables:

$$w(\theta, \bar{\theta}) = \sum_{n_1, n_2, \dots, n_d=0}^{k'-1} [n_1]_q! [n_2]_q! \dots [n_d]_q! \theta_1^{k'-1-n_1} \theta_2^{k'-1-n_2} \dots \theta_d^{k'-1-n_d} \bar{\theta}_1^{k'-1-n_1} \bar{\theta}_2^{k'-1-n_2} \dots \bar{\theta}_d^{k'-1-n_d}. \quad (42)$$

The integration is also carried out in the sense of Berezin-Majid-Rodríguez-Plaza integrals *i.e.* Eqs. (9) and (10) hold for each of the variables θ_i and $\bar{\theta}_i$.

We define d -paragrassmann coherent states as tensor products of d single mode paragrassmann coherent states given in Eq. (20):

$$|\theta\rangle = |\theta_1 \theta_2 \dots \theta_d\rangle = |\theta_1\rangle \otimes |\theta_2\rangle \otimes \dots \otimes |\theta_d\rangle \quad (43)$$

$$= \sum_{n_1, n_2, \dots, n_d=0}^{k'-1} \frac{\theta_1^{n_1} \theta_2^{n_2} \dots \theta_d^{n_d}}{([n_1]_q! [n_2]_q! \dots [n_d]_q!)^{1/2}} |n_1, n_2, \dots n_d\rangle. \quad (44)$$

Together with the measure in (41), these states provide us with a resolution of the unity in the Hilbert space $(\mathbb{C}^{k'})^{\otimes d}$:

$$\int d\theta_d \dots d\theta_1 :: |\theta\rangle w(\theta, \bar{\theta})(\theta) :: d\bar{\theta}_d \dots d\bar{\theta}_1 = I. \quad (45)$$

Here also, because of the noncommutativity of the paragrassmann variables, an ordering should be adopted. In Eq. (45) the $:: ::$ stands for an ordering in which all the θ 's are to the left of the $\bar{\theta}$'s, and the θ 's (as well as the $\bar{\theta}$'s) are ordered according to their indices in increasing order.

Let us use the notation $f(\theta, \bar{\theta})$ for a generic polynomial function of all the variables θ_i and $\bar{\theta}_i$. Then following the quantization scheme described in Section 2, such a generic function can be quantized and mapped to an operator A_f acting on $(\mathbb{C}^{k'})^{\otimes d}$. The corresponding operator is given by

$$A_f = \int d\theta :: f |\theta\rangle w(\theta, \bar{\theta})(\theta) :: d\bar{\theta}, \quad (46)$$

where the following short-handed notations are adopted:

$$d\theta = d\theta_d \dots d\theta_1 \text{ and } d\bar{\theta} = d\bar{\theta}_d \dots d\bar{\theta}_1.$$

For the simplest functions we get the following results:

$$\begin{aligned} A_{\theta_i} &= \int d\theta :: \theta_i |\theta) w(\theta, \bar{\theta})(\theta| :: d\bar{\theta} \\ &= \sum_{n_1, n_2, \dots, n_d=0}^{k'-1} \left([n_i + 1]_q \right)^{1/2} |n_1, n_2, \dots, n_d\rangle \langle n_1, n_2, \dots, n_i + 1, \dots, n_d|, \end{aligned} \quad (47)$$

$$A_{\bar{\theta}_j} = \sum_{n_1, n_2, \dots, n_d=0}^{k'-1} \left([n_j + 1]_q \right)^{1/2} |n_1, n_2, \dots, n_j + 1, \dots, n_d\rangle \langle n_1, n_2, \dots, n_d|. \quad (48)$$

We recognize in A_{θ_i} (respectively, $A_{\bar{\theta}_j}$) the lowering or annihilation operator (respectively, raising or creation operator) in the i^{th} mode. The product of two such operators is given by:

$$\begin{aligned} A_{\theta_i} A_{\bar{\theta}_j} &= \\ &\sum_{n_1, n_2, \dots, n_d=0}^{k'-1} \left([n_i + 1]_q [n_j + 1]_q \right)^{1/2} |n_1, n_2, \dots, n_j + 1, \dots, n_d\rangle \langle n_1, n_2, \dots, n_i + 1, \dots, n_d|. \end{aligned} \quad (49)$$

and

$$\begin{aligned} A_{\bar{\theta}_j} A_{\theta_i} &= \\ &\sum_{n_1, n_2, \dots, n_d=0}^{k'-1} \left([n_i + 1]_q [n_j + 1]_q \right)^{1/2} |n_1, n_2, \dots, n_j + 1, \dots, n_d\rangle \langle n_1, n_2, \dots, n_i + 1, \dots, n_d|. \end{aligned} \quad (50)$$

For $i = j$ we derive a result similar to the one-variable case (28):

$$A_{\theta_i} A_{\bar{\theta}_i} - q A_{\bar{\theta}_i} A_{\theta_i} = \bar{Q}_i, \quad (51)$$

where

$$\bar{Q}_i = \bar{q}^{N_i} = q^{-N_i} = \sum_{n_1, n_2, \dots, n_d=0}^{k'-1} \bar{q}^{n_i} |n_1, n_2, \dots, n_d\rangle \langle n_1, n_2, \dots, n_d|. \quad (52)$$

Eq. (51) is the quantized version of the *q-commutativity* expressed in the last of the equations in (39). The quantization process breaks this *q-commutativity* and replaces it by the non-*q-commutativity* (51).

For $i < j$ the corresponding operators commute

$$A_{\theta_i} A_{\bar{\theta}_j} - A_{\bar{\theta}_j} A_{\theta_i} = 0. \quad (53)$$

In this case, the quantization process broke the *q-commutativity* and replaced it by ordinary commutative relations.

Now let us quantize higher order terms of θ 's:

$$A_{\theta_i \theta_j} = \sum_{n_1, n_2, \dots, n_d=0}^{k'-1} \left([n_i + 1]_q [n_j + 1]_q \right)^{1/2} |n_1, n_2, \dots, n_d\rangle \langle n_1, \dots, n_i + 1, \dots, n_j + 1, \dots, n_d|; \quad (54)$$

this holds for $i \neq j$ and we have:

$$A_{\theta_i \theta_j} = A_{\theta_i} A_{\theta_j} = A_{\theta_j \theta_i} = A_{\theta_j} A_{\theta_i}. \quad (55)$$

This result is similar to Eq. (53): the *q-commutativity* in the first equation in (39) is broken and replaced by *commutativity*. Eq. (55) endows another feature: to, apparently, two different classical functions, $\theta_i \theta_j$ and $\theta_j \theta_i$ there corresponds the same quantum operator $A_{\theta_i \theta_j} = A_{\theta_j \theta_i}$. This feature is a generic consequence of the adopted antinormal and index ordering in (45).

For $i = j$ we have

$$A_{\theta_i \theta_i} = \sum_{n_1, n_2, \dots, n_d=0}^{k'-1} \left(\frac{[n_i + 2]_q!}{[n_i]_q!} \right)^{1/2} |n_1, n_2, \dots, n_d\rangle \langle n_1, \dots, n_i + 2, \dots, n_d|, \quad (56)$$

and we have the same homomorphism for higher order terms of θ and $\bar{\theta}$:

$$A_{\theta_i}^n = A_{\theta_i^n}, \quad A_{\bar{\theta}_i}^n = A_{\bar{\theta}_i^n}. \quad (57)$$

So we have the same nilpotency conditions for the quantized operators associated with paragrassmann variables:

$$A_{\theta_i}^{k'} = A_{\bar{\theta}_i}^{k'} = 0.$$

These operators could be useful for the description of *parafermions* [4] or *k-fermions* [20], which are hypothetical particles obeying a sort of generalized Pauli's exclusion principle: the maximum number of parafermions allowed to occupy the same quantum state is $k - 1$ (1 for fermions).

If one would follow this reasoning and use the d -dimensional paragrassmann algebra together with the associated quantized algebra to describe such particles, the operators $A_{\bar{\theta}_i}$ could be interpreted as the creation operator in the i^{th} mode, while A_{θ_i} would annihilate a particle from the same mode. What we learn from the commutation relations between

the quantized operators is that particles in different modes do not interact with each other; this is the essence of the following commutation relations:

$$\begin{aligned} [A_{\theta_i}, A_{\theta_j}] &= 0, & [A_{\bar{\theta}_i}, A_{\bar{\theta}_j}] &= 0, \\ [A_{\theta_i}, A_{\bar{\theta}_j}] &= 0, \end{aligned} \tag{58}$$

while in the same mode the operators obey the non- q -commutativity relation in (51).

10 Conclusion

In this work we have implemented a coherent-state quantization of the paragrassmann algebra Σ_k viewed as a “classical” phase space in a wide sense. The followed procedure has yielded a unique correspondence between Σ_k and the $k' \times k'$ matrix realization of the so-called k -fermionic algebra F_k [20, 22, 29]. In particular, the q -commutation relations between paragrassmann generators are mapped to the Biedenharn-Macfarlane commutation relations for the q -oscillator [26, 27]. Moreover, the properties of nilpotency and hermiticity of the matrix operators A_θ and $A_{\bar{\theta}}$, which are required in view of the construction of paragrassman coherent states in [22, 21], arise naturally in our construction.

We also note that, as a result of the uniqueness of the correspondence $f(\theta, \bar{\theta}) \mapsto A_f$, one can reexpress any finite-dimensional quantum algebra in terms of the Moyal algebra of the corresponding symbols.

We believe that our approach to quantizing paragrassmann algebras is suitable for the investigation of classical systems of particles with paragrassmann degrees of freedom and their quantum analogs, in the spirit of the Berezin-Marinov particle models [30]. We also think that the displayed one-to-one correspondence between finite-dimensional matrix algebra and a more elaborate algebraic structure sheds a new light on the question of the equivocal nature of what we call quantization. Finally, we think that it would be of great interest to explore such “classical”-quantum correspondence within the quantum information context by extending our formalism to tensor products of quantum states and studying their respective “classical” paragrassmanian counterparts.

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11 Appendix

Commutation relations between first-order elements:

$$\begin{aligned}
A_\theta^L A_\theta^R &= \sum_{n=0}^{k'-1} \left([n+1]_q [n+1]_q \right)^{1/2} |n\rangle \langle n| \\
A_\theta^L A_\theta^L &= A_\theta^R A_\theta^R = \sum_{n=0}^{k'-1} \left([n+1]_q [n+1]_q \right)^{1/2} q_k^{n+2} |n\rangle \langle n| \\
A_\theta^R A_\theta^L &= \sum_{n=0}^{k'-1} \left([n+1]_q [n+1]_q \right)^{1/2} q_k^{2n+4} |n\rangle \langle n| \\
A_\theta^R A_\theta^L &= \sum_{n=0}^{k'-1} \left([n]_q [n]_q \right)^{1/2} |n\rangle \langle n|, \quad A_\theta^R A_\theta^R = A_\theta^L A_\theta^L = \sum_{n=0}^{k'-1} \left([n]_q [n]_q \right)^{1/2} q_k^{n+1} |n\rangle \langle n| \\
A_\theta^L A_\theta^R &= \sum_{n=0}^{k'-1} \left([n]_q [n]_q \right)^{1/2} q_k^{2n+2} |n\rangle \langle n|
\end{aligned}$$

Higher order mixed terms

Now let us consider mixed terms such as $\theta^n \bar{\theta}^m$ with $n > m$.

$$\begin{aligned}
A_{\theta^n \bar{\theta}^m} &= A_{\bar{\theta}^m \theta^n} = \int d\theta |\theta| \theta^n w(\theta, \bar{\theta}) \bar{\theta}^m (\theta) d\bar{\theta} = \sum_{l=0}^{k'-1} \left\{ \frac{[l+n]_q!}{[l]_q!} \frac{[l+n]_q!}{[l+n-m]_q!} \right\}^{1/2} |l\rangle \langle l+n-m| \\
&= A_{\theta^n} A_{\bar{\theta}^m}.
\end{aligned} \tag{59}$$

It is not true, however, that $A_{\theta^n} A_{\bar{\theta}^m} = A_{\bar{\theta}^m} A_{\theta^n}$:

$$A_{\bar{\theta}^m} A_{\theta^n} = \sum_{l=0}^{k'-1} \left\{ \frac{[l+n]_q!}{[l]_q!} \frac{[l+m]_q!}{[l]_q!} \right\}^{1/2} |l+m\rangle \langle l+n|$$

Note that this result is also valid for $n = m$. In the case $m > n$, one has

$$A_{\theta^n \bar{\theta}^m} = A_{\bar{\theta}^m \theta^n} = \sum_{l=0}^{k'-1} \left\{ \frac{[l+m]_q!}{[l+m-n]_q!} \frac{[l+m]_q!}{[l]_q!} \right\}^{1/2} |l+m-n\rangle \langle l| = A_{\theta^n} A_{\bar{\theta}^m}. \tag{60}$$

One can write

$$A_\theta A_{\bar{\theta}^m} = A_\theta A_{\bar{\theta}} A_{\bar{\theta}^{m-1}} = [A_\theta, A_{\bar{\theta}}] A_{\bar{\theta}^{m-1}} + A_{\bar{\theta}} A_\theta A_{\bar{\theta}^{m-1}}$$

and successively, so that

$$[A_\theta, A_{\bar{\theta}^{m-1}}] = \sum_{r=0}^{m-1} A_{\bar{\theta}^r} [A_\theta, A_{\bar{\theta}}] A_{\bar{\theta}^{m-1-r}}$$

which corresponds to total symmetrization of $A_{\bar{\theta}^m}$ around the commutator. By iterating the expression

$$A_{\theta^n} A_{\bar{\theta}^m} = A_{\theta} A_{\theta^{n-1}} A_{\bar{\theta}^m} = A_{\theta} [A_{\theta^{n-1}}, A_{\bar{\theta}^m}] + [A_{\theta}, A_{\bar{\theta}^m}] A_{\theta^{n-1}} + A_{\bar{\theta}^m} A_{\theta^n}$$

one obtains

$$[A_{\theta^n}, A_{\bar{\theta}^m}] = \sum_{r=0}^{n-1} A_{\theta^r} [A_{\theta}, A_{\bar{\theta}^m}] A_{\theta^{n-1-r}} = \sum_{s=0}^{n-1} \sum_{r=0}^{m-1} A_{\theta^s} A_{\bar{\theta}^r} [A_{\theta}, A_{\bar{\theta}}] A_{\bar{\theta}^{m-1-r}} A_{\theta^{n-1-s}}$$

The k -fermionic algebra F_k Quoting section 2.1 from [20], we define here the k -fermionic algebra F_k . “The algebra F_k is spanned by five operators f_- , f_+ , f_+^+ , f_-^+ and N through the following relations classified in three types.

(i) The $[f_-, f_+, N]$ -type:

$$\begin{aligned} f_- f_+ - q f_+ f_- &= 1 \\ N f_- - f_- N &= -f_-, \quad N f_+ - f_+ N = +f_+ \\ (f_-)^k &= (f_+)^k = 0 \end{aligned}$$

(ii) The $[f_+^+, f_-^+, N]$ -type:

$$\begin{aligned} f_+^+ f_-^+ - \bar{q} f_-^+ f_+^+ &= 1 \\ N f_+^+ - f_+^+ N &= -f_+^+, \quad N f_-^+ - f_-^+ N = +f_-^+ \\ (f_+^+)^k &= (f_-^+)^k = 0 \end{aligned}$$

(iii) The $[f_-, f_+, f_+^+, f_-^+]$ -type:

$$f_- f_+^+ - q^{-\frac{1}{2}} f_+^+ f_- = 0, \quad f_+ f_-^+ - q^{+\frac{1}{2}} f_-^+ f_+ = 0$$

where the number

$$q := \exp\left(\frac{2\pi i}{k}\right), \quad k \in \mathbf{N} \setminus \{0, 1\}$$

is a root of unity and \bar{q} stands for the complex conjugate of q . The couple (f_-, f_+^+) of annihilation operators is connected to the couple (f_+, f_-^+) of creation operators via the Hermitean conjugation relations

$$f_+^+ = (f_+)^{\dagger}, \quad f_-^+ = (f_-)^{\dagger}$$

and N is an Hermitean operator. It is clear that the case $k = 2$ corresponds to fermions and the case $k \rightarrow \infty$ to bosons. In the two latter cases, we can take $f_- \equiv f_+^+$ and $f_+ \equiv f_-^+$. In the other cases, the consideration of the two couples (f_-, f_+^+) and (f_+, f_-^+) is absolutely necessary. In the case where k is arbitrary, we shall speak of k -fermions.”

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Erratum: “Coherent state quantization of paragrassmann algebras”

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In the article [1], some of the statements concerning odd paragrassmann algebras are misleading, due to the presence of negative values of the deformations of integers $[n]_q$. We hereby redefine the odd algebras in order to ensure that not only all deformations $[n]_q$ are positive, but also to provide a unified treatment of both even and odd algebras, as can be done in the representation theory of the q -oscillator where q is a primitive root of unity [2].

Changes in section 3 “The essential paragrassmann algebra”. The leading paragraphs must be changed to:

“In this paper, we consider as an observation set X the paragrassmann algebra Σ_k for k even [3, 4, 5, 6]. We recall that a Grassmann (or exterior) algebra of a given vector space V over a field is the algebra generated by the exterior (or wedge) product for which all elements are nilpotent, $\theta^2 = 0$. Paragrassmann algebras are generalizations for which, given an even number $k > 2$, all elements obey $\theta^{k'} = 0$, where $k' = k/2$ for even k . For a given k , we define Σ_k as the linear span of $\{1, \dots, \theta^n, \dots, \theta^{k'-1}\}$ and of their respective *conjugates* $\bar{\theta}^n$: here θ is a paragrassmann variable satisfying $\theta^{k'} = 0$.

Variables θ and $\bar{\theta}$ do not commute:

$$\theta \bar{\theta} = q_k \bar{\theta} \theta, \quad (1)$$

where $q_k = q^2$, $q = e^{\frac{2\pi i}{k}}$. The motivation backing the above definition is to choose the deformation parameter in the paragrassmann algebra as a root of unity of the same order as the nilpotency order: $(q_k)^{k/2} = 1$.

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Thus, the number k fixes the order of the root q and the order of nilpotency of the paragrassmann algebra. The choice of even values of k is necessary to enforce that all deformations of integers $[n]_q$ considered are non-negative, and that the Fock representations of the quantized paragrassmann algebras have dimension k' , and therefore match the representation theory of the q -oscillator algebra when q is a k th root of unity (see, e.g., [7]).”

Changes in section 9 “Generalization to d -dimensional paragrassmann algebra” The line following equation (39) must be “Here q_k is a $\frac{k}{2}$ th root of unity, as before.”

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